SL Paper 1

Let $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be defined by $f(x,\ y) = (x + 3y,\ 2x - y)$.

a. Given that *A* is the interval $\{x: 0 \leqslant x \leqslant 3\}$ and *B* is the interval $\{y: 0 \leqslant x \leqslant 4\}$ then describe *A* × *B* in geometric form. [3]

b.i. Show that the function f is a bijection. [8]

b.ii. Hence find the inverse function f^{-1}. [2]

The group $\{G,\ *\}$ is defined on the set $G = \{1, 2, 3, 4, 5,\ 6\}$ where $*$ denotes multiplication modulo 7.

a. Draw the Cayley table for $\{G,\ *\}$. [3]

b. (i) Determine the order of each element of $\{G,\ *\}$. [6]

(ii) Find all the proper subgroups of $\{G,\ *\}$.

c. Solve the equation $x * 6 * x = 3$ where $x \in G$. [3]

Let G denote the set of 2×2 matrices whose elements belong to $\mathbb{R}$ and whose determinant is equal to 1. Let $*$ denote matrix multiplication which may be assumed to be associative.

Let H denote the set of 2×2 matrices whose elements belong to $\mathbb{Z}$ and whose determinant is equal to 1.

a. Show that $\{G,\ *\}$ is a group. [5]

b. Determine whether or not $\{H,\ *\}$ is a subgroup of $\{G,\ *\}$. [4]

The relations ρ_1 and ρ_2 are defined on the Cartesian plane as follows

$(x_1,\ y_1)\rho_1(x_2,\ y_2) \Leftrightarrow x_1^2 - x_2^2 = y_1^2 - y_2^2$

$(x_1,\ y_1)\rho_2(x_2,\ y_2) \Leftrightarrow \sqrt{x_1^2 + x_2^2} \leqslant \sqrt{y_1^2 + y_2^2}.$

a. For ρ_1 and ρ_2 determine whether or not each is reflexive, symmetric and transitive. [11]

b. For each of ρ_1 and ρ_2 which is an equivalence relation, describe the equivalence classes. [2]

The permutation P is given by

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 2 & 1 \end{pmatrix}.$$

a. Determine the order of P, justifying your answer. [2]

b. Find P^2. [2]

c. The permutation group G is generated by P. Determine the element of G that is of order 2, giving your answer in cycle notation. [4]

The set P contains all prime numbers less than 2500.

The set Q is the set of all subsets of P.

The set S contains all positive integers less than 2500.

The function $f: S \to Q$ is defined by $f(s)$ as the set of primes exactly dividing s, for $s \in S$.
For example $f(4) = \{2\},\ f(45) = \{3,\ 5\}$.

a. Explain why only one of the following statements is true [4]

(i) $17 \subset P$;

(ii) $\{7,\ 17,\ 37,\ 47,\ 57\} \in Q$;

(iii) $\phi \subset Q$ and $\phi \in Q$, where ϕ is the empty set.

b. (i) State the value of $f(1)$, giving a reason for your answer. [4]

(ii) Find $n\left(f(2310)\right)$.

c. Determine whether or not f is [4]

(i) injective;

(ii) surjective.

The group $\{G,\ *\}$ has a subgroup $\{H,\ *\}$. The relation R is defined, for $x,\ y \in G$, by xRy if and only if $x^{-1} * y \in H$.

(a) Show that R is an equivalence relation.

(b) Given that $G = \{0,\ \pm 1,\ \pm 2,\ \ldots\}$, $H = \{0,\ \pm 4,\ \pm 8,\ \ldots\}$ and $*$ denotes addition, find the equivalence class containing the number 3.

G is a group. The elements $a, b \in G$, satisfy $a^3 = b^2 = e$ and $ba = a^2 b$, where e is the identity element of G .

a. Show that $(ba)^2 = e$. [3]

b. Express $(bab)^{-1}$ in its simplest form. [3]

c. Given that $a \neq e$, [6]

(i) show that $b \neq e$;

(ii) show that G is not Abelian.

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a. The set $\mathrm{S}_1 = \{2, 4, 6, 8\}$ and $\times_{10}$ denotes multiplication modulo 10. [8]

(i) Write down the Cayley table for $\{\mathrm{S}_1, \times_{10}\}$.

(ii) Show that $\{\mathrm{S}_1, \times_{10}\}$ is a group.

(iii) Show that this group is cyclic.

b. Now consider the group $\{\mathrm{S}_1, \times_{20}\}$ where $\mathrm{S}_2 = \{1, 9, 11, 19\}$ and $\times_{20}$ denotes multiplication modulo 20. Giving a reason, state whether or not $\{\mathrm{S}_1, \times_{10}\}$ and $\{\mathrm{S}_1, \times_{20}\}$ are isomorphic. [3]

Let $\boldsymbol{S}$ be the set of matrices given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} ; a, b, c, d \in \mathbb{R}, ad - bc = 1$$

The relation R is defined on S as follows. Given $\boldsymbol{A}$, $\boldsymbol{B} \in S$, $\boldsymbol{ARB}$ if and only if there exists $\boldsymbol{X} \in S$ such that $\boldsymbol{A} = \boldsymbol{BX}$.

a. Show that R is an equivalence relation. [8]

b. The relationship between a , b , c and d is changed to $ad - bc = n$. State, with a reason, whether or not there are any non-zero values of n , other than 1, for which R is an equivalence relation. [2]

Consider the set $S = \{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$ under the operation of addition modulo 6, denoted by $+_6$.

a. Construct the Cayley table for $\{S,\ +_6\}$. [2]

b. Show that $\{S,\ +_6\}$ forms an Abelian group. [5]

c. State the order of each element. [2]

d. Explain whether or not the group is cyclic. [2]

Prove that the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $f(x,\ y) = (2x + y,\ x + y)$ is a bijection.

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a. Show that the set S of numbers of the form $2^m \times 3^n$, where $m, n \in \mathbb{Z}$, forms a group $\{S, \times\}$ under multiplication. [6]

b. Show that $\{S, \times\}$ is isomorphic to the group of complex numbers $m + n\text{i}$ under addition, where $m, n \in \mathbb{Z}$. [6]

The relation R is defined on the set $\mathbb{Z}$ by aRb if and only if $4a + b = 5n$, where $a, b, n \in \mathbb{Z}$.

a. Show that R is an equivalence relation. [8]

b. State the equivalence classes of R . [3]

The function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ is defined by $f(x,\ y) = \left(xy,\ \frac{x}{y}\right)$.

Prove that f is a bijection.

The transformations T_1, T_2, T_3, T_4, in the plane are defined as follows:

T_1 : A rotation of 360° about the origin
T_2 : An anticlockwise rotation of 270° about the origin
T_3 : A rotation of 180° about the origin
T_4 : An anticlockwise rotation of 90° about the origin.

The transformation T_5 is defined as a reflection in the x-axis.

The transformation T is defined as the composition of T_3 followed by T_5 followed by T_4.

a. Copy and complete the following Cayley table for the transformations of T_1, T_2, T_3, T_4, under the operation of composition of transformations. [2]

	T_1	T_2	T_3	T_4
T_1	T_1	T_2	T_3	T_4
T_2	T_2			
T_3	T_3			
T_4	T_4			

b.i. *Show that* T_1, T_2, T_3, T_4 under the operation of composition of transformations form a group. Associativity may be assumed. [3]

b.ii. Show that this group is cyclic. [1]

c. Write down the 2×2 matrices representing T_3, T_4 and T_5. [3]

d.i. Find the 2×2 matrix representing T. [2]

d.ii. Give a geometric description of the transformation T. [1]

$\{G, *\}$ is a group of order N and $\{H, *\}$ is a proper subgroup of $\{G, *\}$ of order n.

(a) Define the right coset of $\{H, *\}$ containing the element $a \in G$.

(b) Show that each right coset of $\{H, *\}$ contains n elements.

(c) Show that the union of the right cosets of $\{H, *\}$ is equal to G.

(d) Show that any two right cosets of $\{H, *\}$ are either equal or disjoint.

(e) Give a reason why the above results can be used to prove that N is a multiple of n.

The set S contains the eight matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

where a, b, c can each take one of the values $+1$ or -1 .

a. Show that any matrix of this form is its own inverse. [3]

b. Show that S forms an Abelian group under matrix multiplication. [9]

c. Giving a reason, state whether or not this group is cyclic. [1]

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a. Prove that the number 14641 is the fourth power of an integer in any base greater than 6. [3]

b. For $a, b \in \mathbb{Z}$ the relation aRb is defined if and only if $\frac{a}{b} = 2^k$, $k \in \mathbb{Z}$. [8]

(i) Prove that R is an equivalence relation.

(ii) List the equivalence classes of R on the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

The group $\{G, +\}$ is defined by the operation of addition on the set $G = \{2n | n \in \mathbb{Z}\}$.

The group $\{H, +\}$ is defined by the operation of addition on the set $H = \{4n | n \in \mathbb{Z}\}$

Prove that $\{G, +\}$ and $\{H, +\}$ are isomorphic.

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a. Use the Euclidean algorithm to find $\gcd(162,\ 5982)$. [4]

b. The relation R is defined on $\mathbb{Z}^+$ by nRm if and only if $\gcd(n,\ m) = 2$. [7]

(i) By finding counterexamples show that R is neither reflexive nor transitive.

(ii) Write down the set of solutions of $nR6$.

A sample of size 100 is taken from a normal population with unknown mean μ and known variance 36.

Another investigator decides to use the same data to test the hypotheses H_0: $\mu = 65$, $H1$: $\mu = 67.9$.

a. An investigator wishes to test the hypotheses H_0: $\mu = 65$, H_1: $\mu > 65$. [3]

He decides on the following acceptance criteria:

Accept H_0 if the sample mean $\bar{x} \le 66.5$

Accept H_1 if $\bar{x} > 66.5$

Find the probability of a Type I error.

b.i. She decides to use the same acceptance criteria as the previous investigator. Find the probability of a Type II error. [3]

b.ii. Find the critical value for $\bar{x}$ if she wants the probabilities of a Type I error and a Type II error to be equal. [3]